4.2 The Great Polynomial Divide

Polynomial Division

LEARNING GOALS

In this lesson, you will:

- Describe similarities between polynomials and integers.
- Determine factors of a polynomial using one or more roots of the polynomial.
- Determine factors through polynomial long division.
- Compare polynomial long division to integer long division.

KEY TERMS

- polynomial long division
- synthetic division

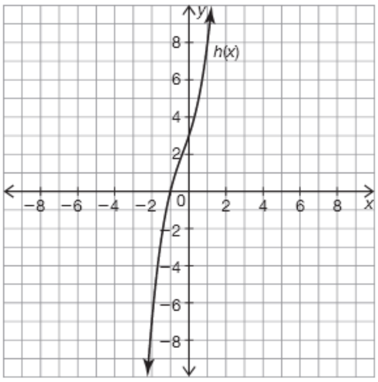
Did you ever notice how little things can sometimes add up to make a huge difference? Consider something as small and seemingly insignificant as a light bulb. For example, a compact fluorescent lamp (CFL), uses less energy than "regular" bulbs. Converting to CFLs seems like a good idea, but you might wonder: how much good can occur from changing one little light bulb? The answer is a ton—especially if you convince others to do it as well. According to the U.S. Department of Energy, if each home in the United States replaced one light bulb with a CFL, it would have the same positive environmental effect as taking 1 million cars off the road!

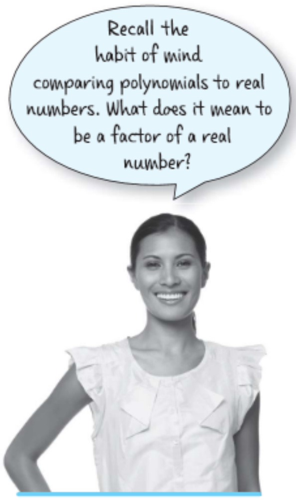
If an invention such as the CFL can have a dramatic impact on the environment, imagine the effect that larger inventions can have. A group of Canadian students designed a car that gets over 2,500 miles per gallon, only to have their invention topped by a group of French students whose car gets nearly 7,000 miles per gallon! What impacts can you describe if just 10% of the driving population used energy efficient cars? Can all of these impacts be seen as positive impacts?

PROBLEM 1 A Polynomial Divided . . .

The previous function-building lessons showed how the factors of a polynomial determine its key characteristics. From the factors, you can determine the type and location of a polynomial's zeros. Algebraic reasoning often allows you to reverse processes and work backwards. Specifically in this problem, you will determine the factors from one or more zeros of a polynomial from a graph.

1. Analyze the graph of the function $h(x) = x^3 + x^2 + 3x + 3$.





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a. Describe the number and types of zeros of $h(x)$.

b. Write the factor of $h(x)$ that corresponds to the zero at $x = -1$.

c. What does it mean to be a factor of $h(x) = x^3 + x^2 + 3x + 3$?

d. How can you write any zero, r, of a function as a factor?

In Question 1 you determined that $(x + 1)$ is a factor of $h(x)$. One way to determine another factor of $h(x)$ is to analyze the problem algebraically through a table of values.

2. Analyze the table of values for $d(x) \cdot q(x) = h(x)$.

x	$d(x) = (x + 1)$	$q(x)$	$h(x) = x^3 + x^2 + 3x + 3$
−3	−2		−24
−2	−1		−7
−1	0		0
0	1		3
1	2		8
2	3		21

a. Complete the table of values for $q(x)$. Explain your process to determine the values for $q(x)$.

b. Two students, Tyler and McCall, disagree about the output $q(-1)$.

Tyler

The output value $q(-1)$ can be any integer. I know this because $d(-1) = 0$. Zero times any number is 0, so I can complete the table with any value for $q(-1)$.

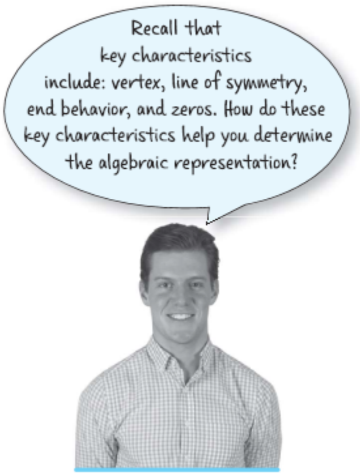
McCall

I know $q(x)$ is a function so only one output value exists for $q(-1)$. I have to use the key characteristics of the function to determine that exact output value.

Who is correct? Explain your reasoning, including the correct output value(s) for $q(-1)$.

c. How can you tell from the table of values that $d(x)$ is a factor of $h(x)$?

3. Describe the key characteristics of $q(x)$. Explain your reasoning.



4. What is the algebraic representation for $q(x)$? Verify algebraically that $d(x) \cdot q(x)$ is equivalent to $h(x)$.

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You can use a graphing calculator to determine the quadratic, cubic, or quartic regression of a set of data.

Step 1: Press **ENTER** to return to the Plot Menu. Scroll down to **7:QUIT**. Press **ENTER**.

Step 2: You will see on the screen Lists 1 through 4 (**L1-L4**). The data needed to determine the quadratic, cubic, or quartic regression is located in Lists 1 and 2 which measures time and distance.

Step 3: Press **STAT**. Scroll right to **CALC**. Choose **5:QuadReg**. Press **ENTER**.

Step 4: Select **L1**, **L2**. Press **ENTER**.

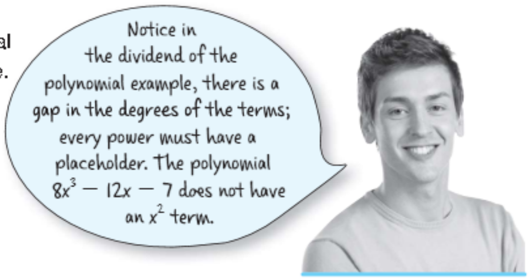
Step 5: The information shows you the standard form of a quadratic, cubic, or quartic equation and the values of a, b, c, and r2.

5. Determine the zeros of $q(x)$. Then rewrite $h(x)$ as a product of its factors.

PROBLEM 2 Long Story Not So Short

The Fundamental Theorem of Algebra states that every polynomial equation of degree n must have n roots. This means that every polynomial can be written as the product of n factors of the form $(ax + b)$. For example, $2x^2 - 3x - 9 = (2x + 3)(x - 3)$. You know that a factor of an integer divides into that integer with a remainder of zero. This process can also help determine other factors. For example, knowing 5 is a factor of 115, you can determine that 23 is also a factor since $\frac{115}{5} = 23$. In the same manner, factors of polynomials also divide into a polynomial without a remainder. Recall that $a \div b$ is $\frac{a}{b}$, where $b \neq 0$.

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division.



Integer Long Division	Polynomial Long Division	Description
$4027 \div 12$ or $\frac{4027}{12}$	$(8x^3 - 12x - 7) \div (2x + 3)$ or $\frac{8x^3 - 12x - 7}{2x + 3}$	
335 12 ⟌ 4027 36 42 36 67 60 (7) Remainder	(B) $4x^2$ − (E) $6x$ + (H) 3 $2x + 3$ ⟌ (C) $8x^3$ + (A) $0x^2 - 12x - 7$ $-(8x^3 + 12x^2)$ ↓ (D) (F) $-12x^2 - 12x$ $-(-12x^2 - 18x)$ ↓ (G) (I) $6x - 7$ $-(6x + 9)$ Remainder (−16)	A. Rewrite the dividend so that each power is represented. Insert $0x^2$. B. Divide $\frac{8x^3}{2x} = 4x^2$. C. Multiply $4x^2(2x + 3)$, and then subtract. D. Bring down $-12x$. E. Divide $\frac{-12x^2}{2x} = -6x$. F. Multiply $-6x(2x + 3)$, and then subtract. G. Bring down -7. H. Divide $\frac{6x}{2x} = 3$. I. Multiply $3(2x + 3)$, and then subtract.
$\frac{4027}{12} = 335$ R 7	$\frac{8x^3 - 12x - 7}{2x + 3} =$ $4x^2 - 6x + 3$ R -16	Rewrite
$4027 = (12)(335) + 7$	$8x^3 - 12x - 7 =$ $(2x + 3)(4x^2 - 6x + 3) - 16$	Check

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1. Analyze the worked example that shows integer long division and polynomial long division.

   a. In what ways are the integer and polynomial long division algorithms similar?

   b. Is $2x + 3$ a factor of $f(x) = 8x^3 - 12x - 7$? Explain your reasoning.

2. Determine the quotient for each. Show all of your work.

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   a. $x\overline{)4x^3 - 0x^2 + 7x}$

   b. $x - 4\overline{)x^3 + 2x^2 - 5x + 16}$

   c. $(4x^4 + 5x^2 - 7x + 9) \div (2x - 3)$

   d. $(9x^4 + 3x^3 + 4x^2 + 7x + 2) \div (3x + 2)$

3. Consider Question 2 parts (a) through (d) to answer each.

   a. Why was the term $0x^2$ included in the dividend in part (a)? Why was this necessary?

   b. When there was a remainder, was the divisor a factor of the dividend? Explain your reasoning.

   c. Describe the remainder when you divide a polynomial by a factor.

4. Determine whether $m(x) = 2x + 1$ is a factor of each function. Explain your reasoning.

   a. $j(x) = 2x^3 + 3x^2 + 7x + 5$

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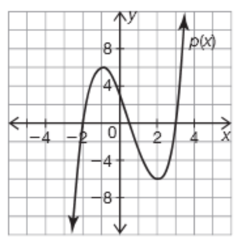
   b.



c.

x	$k(x)$
−2	−9
−1	−4
0	5
1	18
2	35

5. Determine the unknown in each.

a. $\frac{x}{7} = 18$ R 2. Determine x.

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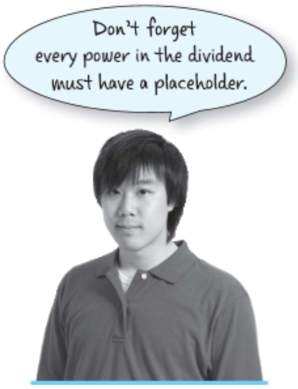
b. $\frac{p(x)}{x + 3} = 3x^2 + 14x + 15$ R 3. Determine the function $p(x)$.

c. Describe the similarities and differences in your solution strategies.

d. Use a graphing calculator to analyze the graph and table of $\frac{p(x)}{x + 3}$ over the interval (−10, 10). What do you notice?

6. Calculate the quotient using long division. Then write the dividend as the product of the divisor and the quotient plus the remainder.



a. $f(x) = x^2 - 1$

$g(x) = x - 1$

Calculate $\frac{f(x)}{g(x)}$.

b. $f(x) = x^3 - 1$

$g(x) = x - 1$

Calculate $\frac{f(x)}{g(x)}$.

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c. $f(x) = x^4 - 1$

$g(x) = x - 1$

Calculate $\frac{f(x)}{g(x)}$.

d. $f(x) = x^5 - 1$

$g(x) = x - 1$

Calculate $\frac{f(x)}{g(x)}$.

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7. Analyze the table of values. Then determine if $q(x)$ is a factor of $p(x)$. If so, explain your reasoning. If not, determine the remainder of $\frac{p(x)}{q(x)}$. Use the last column of the table to show your work.

x	$q(x)$	$p(x)$	
0	1	5	
1	2	7	
2	3	9	
3	4	11	
4	5	13	

8. Look back at the various polynomial division problems you have seen so far. Do you think polynomials are closed under division? Explain your reasoning.

PROBLEM 3 Improve Your Efficiency Rating

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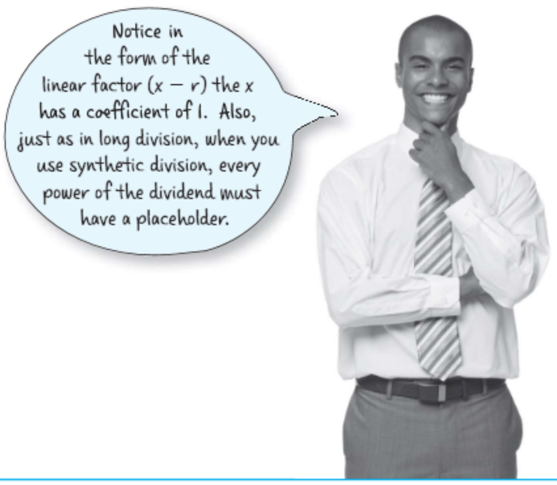
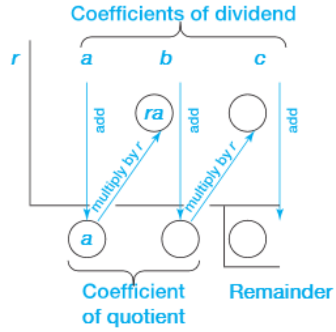
Although dividing polynomials through long division is analogous to integer long division, it can still be inefficient and time consuming. **Synthetic division** is a shortcut method for dividing a polynomial by a linear factor of the form $(x - r)$. This method requires fewer calculations and less writing by representing the polynomial and the linear factor as a set of numeric values. After the values are processed, you can then use the numeric outputs to construct the quotient and the remainder.



To use synthetic division to divide a polynomial $ax^2 + bx + c$ by a linear factor $x - r$, follow this pattern.



You can use synthetic division in place of the standard long division algorithm to determine the quotient for $(2x^2 - 3x - 9) \div (x - 3)$.

Long Division	Synthetic Division

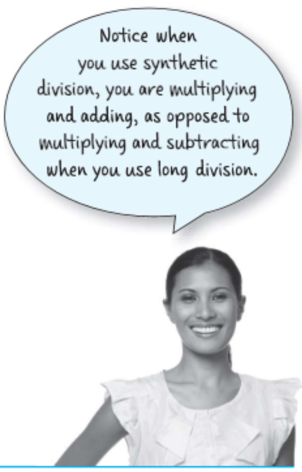
$2x + 3$
$x - 3\overline{)2x^2 - 3x - 9}$
$\underline{2x^2 - 6x}$
$3x - 9$
$\underline{3x - 9}$
0
	$r = 3$
	3 │ 2 −3 −9
	        6 9
	   2 3 │ 0
	(add; multiply by r)

$(2x^2 - 3x - 9) \div (x - 3) = 2x + 3$	

1. Analyze the worked example.

   a. Write the dividend as the product of its factors.

   b. Why does the synthetic division algorithm work?



2. Two examples of synthetic division are provided. Perform the steps outlined for each problem:

i. Write the dividend.

ii. Write the divisor.

iii. Write the quotient.

iv. Write the dividend as the product of the divisor and the quotient plus the remainder.

a.

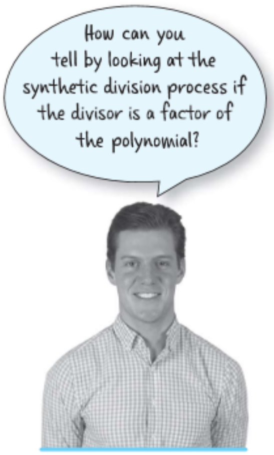
2	1	0	−4	−3	6
		2	4	0	−6
	1	2	0	−3	0

i.

ii.

iii.

iv.



b.

−3	2	−4	−4	−3	6
		−6	30	−78	243
	2	−10	26	−81	249

i.

ii.

iii.

iv.

3. Calculate each quotient using synthetic division. Then write the dividend as the product of the divisor and the quotient plus the remainder.

a. $g(x) = x^3 + 1$

$r(x) = x + 1$

Calculate $\frac{g(x)}{r(x)}$.

b. $g(x) = x^3 + 8$

$r(x) = x + 2$

Calculate $\frac{g(x)}{r(x)}$.

c. $g(x) = x^3 + 27$

$r(x) = x + 3$

Calculate $\frac{g(x)}{r(x)}$.

d. $g(x) = x^3 + 64$

$r(x) = x + 4$

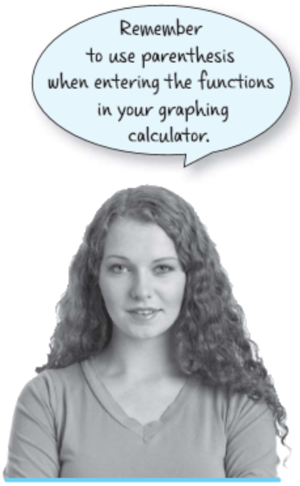
Calculate $\frac{g(x)}{r(x)}$.

4. Use a graphing calculator to compare the graphs and table of values for each pair of functions.

Group 1: $g(x) = \dfrac{x^3 + 1}{x + 1}$ and $j(x) = x^2 - x + 1$

Group 2: $g(x) = \dfrac{x^3 + 8}{x + 2}$ and $j(x) = x^2 - 4x + 4$

Group 3: $g(x) = \dfrac{x^3 + 27}{x + 3}$ and $j(x) = x^2 - 9x + 3$



a. Describe the similarities and differences in the graphs and tables of values within each pair of functions.

b. Are the functions within each pair equivalent? Explain your reasoning.

Synthetic division works only for linear divisors in the form $x - r$. If the divisor has a leading coefficient other than 1, you may need to factor out a constant in order to rewrite the divisor in the form $x - r$.

You can use synthetic division to determine the quotient of $\frac{2x^3 - 6x^2 + 4x + 2}{2x - 3}$. Since the divisor is not in the form $x - r$, you can rewrite $2x - 3$ as $2\left(x - \frac{3}{2}\right)$.

$r = \frac{3}{2}$

$\frac{3}{2}$	2	−6	4	2
		3	$-\frac{9}{2}$	$-\frac{3}{4}$
	2	−3	$-\frac{1}{2}$	$\frac{5}{4}$

(add; multiply by r)

The numbers in the last row become the coefficients of the quotient.

$2x^2 - 3x - \frac{1}{2}$ R $\frac{5}{4}$

You can write the dividend as the product of the divisor and the quotient plus the remainder.

$$2x^3 - 6x^2 + 4x + 2 = \left(x - \frac{3}{2}\right)\left(2x^2 - 3x - \frac{1}{2}\right) + \frac{5}{4}$$

5. Verify $(3x - 2)(x^2 + x + 1) = 3x^3 + x^2 + x - 2$ using synthetic division. Show all work and explain your reasoning.

6. Analyze each division problem given $f(x) = x^3 - 3x^2 - x + 3$.

$$g(x) = \frac{f(x)}{x - 1} \qquad h(x) = \frac{f(x)}{2x - 2} \qquad j(x) = \frac{f(x)}{3x - 3}$$

a. Determine the quotient of each function.

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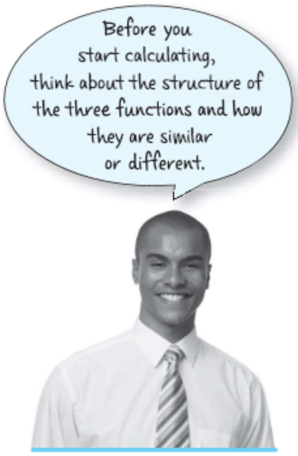
b. Use function notation to write $h(x)$ and $j(x)$ in terms of $g(x)$.

c. Use a graphing calculator to compare the graphical representations of $g(x)$, $h(x)$, and $j(x)$. What are the similarities and differences in the key characteristics? Explain your reasoning.



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d. Given the function $g(x)$, describe the transformation(s) that occurred to produce $h(x)$ and $j(x)$.

7. Is the function $q(x) = (x + 2)$ a factor of the function $p(x) = (x + 2)(x - 4)(x + 3) + 1$? Show all work and explain your reasoning.

8. The lesson opener discussed efficiency. Describe patterns and algorithms learned in this lesson that made your mathematical work more efficient.

Be prepared to share your solutions and methods.